Optimal growth trajectories with finite carrying capacity

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We consider the problem of finding optimal strategies that maximize the average growth rate of multiplicative stochastic processes. For a geometric Brownian motion, the problem is solved through the so-called Kelly criterion, according to which the optimal growth rate is achieved by investing a constant given fraction of resources at any step of the dynamics. We generalize these finding to the case of dynamical equations with finite carrying capacity, which can find applications in biology, mathematical ecology, and finance. We formulate the problem in terms of a stochastic process with multiplicative noise and a nonlinear drift term that is determined by the specific functional form of carrying capacity. We solve the stochastic equation for two classes of carrying capacity functions (power laws and logarithmic), and in both cases we compute the optimal trajectories of the control parameter. We further test the validity of our analytical results using numerical simulations.

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I. INTRODUCTION

There are many interesting connections between statistical mechanics, biology [1], and the theory of financial markets [2,3]. For instance, it has been stressed recently that the lack of ergodicity in the geometric Brownian motion process [4] has important implications for the optimal leverage problem, i.e., the problem of finding how much of a portfolio should be reinvested over time to maximize the logarithmic growth rate of capital [5].

For multiplicative processes, such as the geometric Brownian motion, the effective growth rate is not given by the drift term alone. More precisely, consider the process described by

$$dK(t) = \rho K(t)[\mu\, dt + \sigma\, dW(t)], \tag{1}$$

with μ the drift of the stochastic process, σ the noise amplitude, and ρ a positive constant. Here $K(t)$ represents the capital of an investor at time t, while ρ is the fraction of capital that is invested in a risky security, also known as the leverage. Using Ito's formula, it can be shown that $\langle dK(t)/K(t)\rangle = \mu\rho\, dt$, while $\langle d\log[K(t)]\rangle = \rho(\mu - \frac{\sigma^2}{2})dt$, where $\langle\cdot\rangle$ represents the ensemble average over the stochastic process $dW(t)$. The fact that these two expected values are different has been interpreted in [5–7] as a characteristic signature of the absence of ergodicity, since the first expression can be identified as an ensemble average while the second can be seen as the time average over an infinitely long single instance of the stochastic process [8].

In this context, it has been argued that maximizing the expected log-return of $K(t)$, often called the Kelly criterion [9–17], is a better objective in the long run than simply maximizing its average. For the case of the geometric Brownian motion, in [5] Peters discusses the differences between these two objectives and provides a new interpretation of the optimal leverage obtained from the Kelly criterion by Thorp [14], and shows that it is given by

$$\rho_{\text{Thorp}} = \frac{\mu}{\sigma^2}. \tag{2}$$

In this paper, we extend this analysis by computing optimal growth trajectories in the case of a multiplicative random process with finite carrying capacity, when the drift term $\hat{\mu} \equiv \mu(\rho K)$ is a decreasing function of $\rho(t)K(t)$. A finite carrying capacity can be associated with the presence of market frictions, such as transaction costs, or in biology as an environment with finite resources. Although we frame it in terms of investment decisions, our analysis is of interest beyond finance. For instance, multiplicative stochastic processes with finite carrying capacity are commonly used in biology to describe the growth of a population constrained by a finite amount of resources and in a random environment. Stochastic Gompertzian differential equations are well known, for instance, in population ecology [18] and cell growth, where one might want to control the amount or resources in order to optimize the growth of a population. One example of such a situation is the logistic equation

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)}{\tilde{K}(t)}\right), \tag{3}$$

where a is a constant and $\tilde{K}(t)$ represents the environmental total resources, and which can be controlled as a function of time. This problem can be approached using the methodology developed in this paper in the more general case in which fluctuations are present. For instance, our methodology applies when considering the case $\tilde{K}(t) = \frac{\tilde{K}}{\rho(t)}$, where we assume that $\tilde{K}$ is a constant that represent the amount of resources available. We compute the optimal parameter $\rho(t)$ for logarithmic and power-law functional forms of $\hat{\mu} \equiv \mu(\rho K)$ for the slightly more complicated case of a continuous reinvestment problem. Specifically, we will exactly solve the two models and evaluate the optimal strategy, i.e., the parameter $\rho(t)$. In addition, we provide a methodology for evaluating the optimal control parameter $\rho(t)$ for generic series expansions of the carrying capacity parameter. Finally, we show that numerical simulations agree with our analytical results.

II. THE MODEL

Consider the following process: At time $t=0$, an investor has $K(0)=K_0$, and, at any discrete time step t, she must decide the fraction $\rho(t)$ of her capital to invest in a risky asset. At the end of each period, the risky asset pays a return $r(t)$, which is drawn from a Gaussian distribution with average μ and standard deviation σ. Here we assume that there is a transaction cost $c(t)$ per dollar associated with the purchase of the risky asset, and that the asset purchased at time t cannot be carried over to the next period, but it needs to be sold at the end of each period. This is inspired by possible applications to wholesale electricity markets [19], in which trades have to be closed at the end of each trading day. Furthermore, we assume that the remaining fraction, $1-\rho(t)$, of the capital is not invested, and that its value does not change over the day (i.e., we assume the risk-free interest rate is 0 for simplicity).

Under the specifications above, the capital evolves between time t and $t+1$ as

$$K(t+1)=\{[r(t)-c(t)]\rho(t)+1\}K(t). \tag{4}$$

If we are interested in the evolution of wealth over time horizons that are much longer than a trading period, we can consider the process in the limit of continuous time:

$$dK(t)=\rho(t)K(t)[r(t)-c(t)]. \tag{5}$$

We now assume that the returns evolve according to the stochastic process

$$r(t)=\mu\, dt+\sigma\, dW_t, \tag{6}$$

and we define $c(t)=f(\frac{\rho(t)K(t)}{\tilde{K}})dt$ and introduce the carrying capacity quantity $\tilde{K}$. We further assume that $f(0)=0$, such that for $\tilde{K}\to\infty$ the transaction cost vanishes. In this case, we can write

$$\frac{dK(t)}{K(t)}=\rho(t)\hat{\mu}(\rho(t)K(t)/\tilde{K},t)dt+\rho(t)K(t)\sigma\, dW_t, \tag{7}$$

where

$$\hat{\mu}(\rho(t)K(t),t)=\mu\left[1-f\left(\frac{\rho(t)K(t)}{\tilde{K}}\right)\right]. \tag{8}$$

Here $\tilde{K}$ is the carrying capacity of the system, which in our context is associated with the cost of purchasing risky assets (see Fig. 1).

A. Analytical solution for $f(x)=x^\gamma$

In this section, we solve the stochastic equation (7) in the case in which $f(x)=x^\gamma$ and $\rho(t)=\rho$ is constant in time. We also use the simplifying assumption that the parameters μ, σ, and γ are constant in time. In this case, we have that

$$\hat{\mu}(\rho K,t)=\mu\left[1-\left(\frac{\rho K}{\tilde{K}}\right)^\gamma\right], \tag{9}$$

where from now on we suppress the argument t for the capital K. If we take $\rho=\tilde{K}=1$, we can write

$$dK=\mu K(1-K^\gamma)dt+\sigma K\, dW_t. \tag{10}$$

Equation (10) can be solved using standard methods [20]. Here we report only the solution for $\gamma=1$, but a full derivation for generic γ can be found in Appendix A 1:

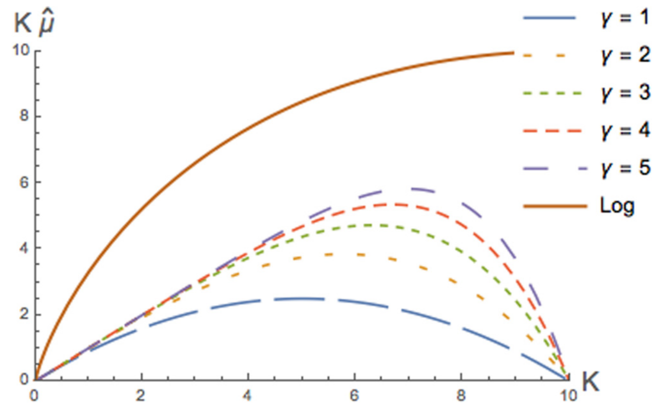

FIG. 1. Return functional $K\hat{\mu}(K)$ of Eq. (8) for $f(x)=x^\gamma$, with $\gamma=1,\ldots,5$, and $f(x)=\log(x)$. The other constants are fixed at $\mu=\rho=1$, $\tilde{K}=10$, and $\alpha=1$.

$$K(t)=\frac{\tilde{K}}{\rho}\, e^{(\rho\mu-\frac{(\rho\sigma)^2}{2})t+\rho\sigma W_t} \times\left(\frac{\tilde{K}}{\rho K_0}+\rho\mu\int_0^t e^{(\rho\mu-\frac{(\rho\sigma)^2}{2})s+\rho\sigma W_s}ds\right)^{-1}. \tag{11}$$

The asymptotic stochastic equilibrium of the above solution can be determined by solving

$$\langle d\log[K(t)/K_0]\rangle=0. \tag{12}$$

In fact, from Eq. (7) and using Itô's theorem, we obtain

$$0=\langle d\log(K/K_0)\rangle=\rho(t)\mu\left[1-f\left(\frac{\rho(t)K(t)}{\tilde{K}}\right)\right]-\frac{\sigma^2}{2}\rho^2, \tag{13}$$

and for linear functions $f(\cdot)$ we obtain $K=\tilde{K}(\frac{1}{\rho}-\frac{\sigma^2}{2\mu})=\tilde{K}(\frac{1}{\rho}-\frac{1}{2\rho_{\text{Thorp}}})$. If the parameter ρ is constant, the above implies an asymptotic equilibrium state $K(\infty)$. This equilibrium point is compatible with our simulations presented in the following. This also holds in the case of the geometrical Brownian motion, where $\rho_{\text{opt}}\equiv\rho_{\text{Thorp}}=\frac{\mu}{\sigma^2}$ was computed in [5,14].

B. Analytical solution for $f(x)=\alpha\log(x)$

In this section, we provide a solution for the logarithmic functional form of carrying capacity, introduced recently in [21,22] for the case of stock markets and describing cell growth [1,23,24]. The equation of interest is

$$dK=\mu K\left[1-\alpha\log\left(\frac{K}{\tilde{K}}\right)\right]dt+\sigma K\, dW_t \tag{14}$$

for the case in which ρ is constant (in Fig. 1 we plot the return functional). The solution is obtained in Appendix A 2, where we find

$$K(t)=\exp\left\{e^{-\alpha\mu t}\left[C_0+\left(\mu[1+\alpha\log(\tilde{K})]-\frac{\sigma^2}{2}\right)\times\frac{e^{\alpha\mu t}-1}{\alpha\mu}+\sigma\int_0^t e^{\alpha\mu s}dW_s\right]\right\}. \tag{15}$$

The expectation $\langle\log[K(t)]\rangle$ can then be evaluated analytically using $\langle\int_0^t f(s)dW_s\rangle = 0$ for any deterministic and continuous function $f(s)$, and thus substituting $C_0 = \log(K_0)$,

$$\langle\log[K(t)]\rangle = e^{-\alpha\mu t}\left\{C_0 + \left[\left(\frac{1}{\alpha} + \log(\tilde{K})\right) - \frac{\sigma^2}{2\alpha\mu}\right] \times (e^{\alpha\mu t} - 1)\right\}. \tag{16}$$

Looking at the asymptotic stochastic equilibrium point again, we have

$$\langle\log[K(t\to\infty)]\rangle = \log(\tilde{K}) + \frac{1}{\alpha}\left(1 - \frac{\sigma^2}{2\mu}\right). \tag{17}$$

We note that while the expected return and the exponential of the expected log-return are not equal, they converge asymptotically.

III. OPTIMAL TRAJECTORIES

We now turn to the problem of finding the optimal trajectory $\rho(t)$ following the Kelly strategy that maximizes the expected log-return of the investor's capital, i.e., $\langle\log[K(t)/K_0]\rangle$. In general, we must solve the equation

$$d\log\left(\frac{K}{K_0}\right) = \left\{\mu\rho(t)\left[1 - f\left(\frac{\rho(t)K(t)}{\tilde{K}}\right)\right] - \rho^2(t)\frac{\sigma^2}{2}\right\}dt + \rho(t)\sigma\, dW_t, \tag{18}$$

which implies

$$\left\langle d\log\left(\frac{K}{K_0}\right)\right\rangle = \left\langle\mu\rho(t)\left[1 - f\left(\frac{\rho(t)K(t)}{\tilde{K}}\right)\right] - \rho^2(t)\frac{\sigma^2}{2}\right\rangle dt + \langle\rho(t)\sigma dW_t\rangle. \tag{19}$$

It is instructive to discuss first the case of a time-dependent ρ in Eq. (19), and then provide approximations for which we can obtain an explicit solution for the optimal parameter ρ. Using (19), the expected logarithmic return over a time horizon T is

$$\log\left(\frac{K(T)}{K_0}\right) = \int_0^T \left\langle\mu\rho(t)\left[1 - f\left(\frac{\rho(t)K(t)}{\tilde{K}}\right)\right] - \rho^2(t)\frac{\sigma^2}{2}\right\rangle dt. \tag{20}$$

The main difference with respect to the case of no carrying capacity is the fact that due to the effective dependence of the drift term on K, optimizing Eq. (20) requires taking a functional derivative with respect to $\rho(t)$ and setting it to zero, i.e.,

$$\frac{\delta}{\delta\rho(t)}\log\left(\frac{K(T)}{K_0}\right) = 0. \tag{21}$$

In the case of a pure geometric Brownian motion, it can be shown that the optimal $\rho(t)$ is constant in time. In the case under consideration, one strategy is to first obtain a solution for arbitrary $\rho(t)$, and then take the functional derivative in the integral of Eq. (20). Unfortunately, this is difficult, so in the following we will resort to two different approximations. First we take the stationary case, in which the solution for *constant* ρ is known, as discussed in Sec. III A. For the second approximation, if we write the function of the carrying capacity as a series expansion, we can write a dynamical set of equations for the derivatives of all the moments of the solution $\langle K(t)^n\rangle$, which then can be optimized iteratively; this procedure will be discussed in Sec. III B.

A. Stationary approximation

In this section, we will use the exact solutions obtained in Secs. II A and II B under the assumption of constant parameter ρ within a quasistationary approximation scheme. More precisely, we assume that $\rho(t)$ is a slow variable with respect to $K(t)$, and that the latter quickly relaxes to what would be its asymptotic value should ρ remain constant. The validity of the approximation can then be assessed from the obtained solution by checking whether $|\langle K(t)\rangle\frac{\partial_t\rho(t)}{\partial_t\langle K(t)\rangle}| \ll 1$.

In general, we have that

$$\begin{aligned}\frac{\delta}{\delta\rho(t)}\log\left(\frac{K(T)}{K_0}\right) &= \frac{\delta}{\delta\rho(t)}\int_0^T\left[\mu\rho(t)\left\{1-\left\langle f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right\rangle\right]-\rho^2(t)\frac{\sigma^2}{2}\right\}dt\\ &= \int_0^T\left\{\mu\left[1 - f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right] - 2\rho(t)\frac{\sigma^2}{2} - \rho(t)\mu\frac{\delta}{\delta\rho(t)}\left\langle f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right\rangle\right\}dt.\end{aligned} \tag{22}$$

To find an optimal solution, we now impose the following condition:

$$\mu\left[1 - f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right] - 2\rho(t)\frac{\sigma^2}{2} - \rho(t)\mu\frac{\delta}{\delta\rho(t)}\left\langle f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right\rangle = 0, \tag{23}$$

where the last functional derivative requires knowledge of $K(\rho(t),t)$ for arbitrary $\rho(t)$. To evaluate this functional derivative, as a first approximation, we will assume that $\rho(t) \approx \rho$ in the interval $\Theta = [t,t+\delta t]$. If this is true, then we can evolve in the interval Θ the solution with a constant ρ from $t_0 = t$ to $t_f = t + \delta t$ with initial condition $K_0 = K(t)$. Let us call such a solution $\mathcal{K}(\delta t,\rho,K(t))$, which satisfies the property $\lim_{\delta t\to 0}\mathcal{K}(\delta t,\rho,K(t)) = K(t)$. The underlying assumption of this approach is that $\rho(t)$ changes slowly with respect to the stochastic dynamics, which implies that $|\langle K(t)\rangle\frac{\partial_t\rho(t)}{\partial_t\langle K(t)\rangle}| \ll 1$. Within this approximation, we can write the functional derivative as

$$\frac{\delta}{\delta\rho(t)}\left\langle f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right\rangle \approx \partial_\rho\left\langle f\left(\rho\frac{\mathcal{K}(t+\delta t,\rho,K(t))}{\tilde{K}}\right)\right\rangle. \tag{24}$$

Then the optimal instantaneous parameter $\rho(t)$ can be obtained by solving the following equation:

$$\mu\left[1 - f\left(\rho(t)\frac{K(t)}{\tilde{K}}\right)\right] - 2\rho(t)\frac{\sigma^2}{2} = \lim_{\delta t\to 0}\rho(t)\mu\,\partial_\rho\left\langle f\left(\rho(t)\frac{\mathcal{K}(\delta t,\rho,K(t))}{\tilde{K}}\right)\right\rangle, \tag{25}$$

which is the approximation we use in the following.

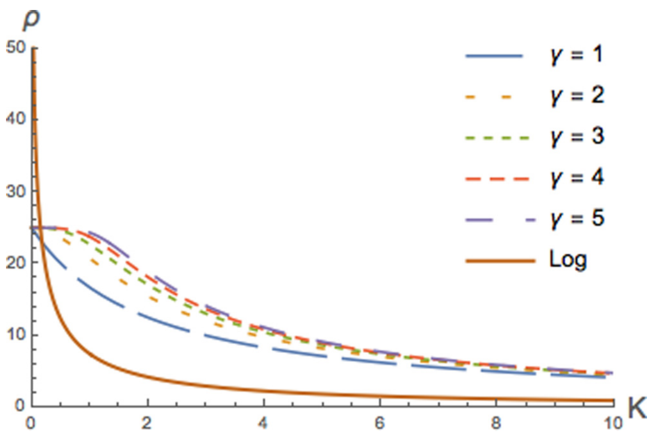

FIG. 2. Optimal parameter $\rho(t)$ obtained numerically from Eq. (26) as a function of $K \in [0,10]$ for various values of γ, and for logarithmic carrying capacity with $\alpha = 1$. Other constants are fixed at $\mu = 1$, $\sigma = 0.2$, and $\tilde{K} = 50$.

1. $f(x) = x^\gamma$: Expansion assuming $\frac{K}{\tilde{K}} \ll 1$

In the case of $f(x) = x^\gamma$, evaluating $\langle f \rangle$ is a nontrivial task. Even assuming we have the stationary approximation, expanding Eq. (A9) in $K/\tilde{K}$, and considering only the zeroth-order term of this expansion, we obtain the following expression to be solved for ρ:

$$\mu\left[1 - \left(\frac{K\rho}{\tilde{K}}\right)^\gamma\right] - \rho\sigma^2 = 0, \tag{26}$$

which cannot be solved analytically for arbitrary values of γ. However, for $K \gg \tilde{K}$, we can obtain the approximate solution

$$\rho(K \gg \tilde{K}) \approx \frac{\tilde{K}}{K}, \tag{27}$$

which is independent of γ. A plot with the numerical solutions of $\rho(K)$ obtained from (26) for different values of γ is shown in Fig. 2.

In the particular case of $\gamma = 1$, the solution is simply

$$\rho(t) = \frac{\mu}{2\mu\frac{K(t)}{\tilde{K}} + \sigma^2}, \tag{28}$$

which shows explicitly that the presence of carrying capacity effectively increases the risk, as the optimal fraction of resources to be deployed is a decreasing function of K.[1] This is important as is it a generalization of the result of [4], and it has direct applications to the problem of optimal trajectories in the context of financial time series where one has an embedded transaction cost. This applies, for instance, to lotteries and wholesale electricity markets, where one has independent processes at each time step. In this case, the transaction cost plays the role of market impact.

Surprisingly, the optimal parameter of Eq. (28) holds up to second order in $\xi = \frac{K}{\tilde{K}}$. To obtain precise estimates of the parameter ρ, the expectation $\langle K \rangle$ must be evaluated. This is done up to second order in ξ in Appendix B using techniques partly developed in [25].

To check the consistency of the stationarity approximation, we evaluate

$$\partial_t \rho(t) = \partial_t \frac{\mu}{2\frac{K(t)}{\tilde{K}} + \sigma^2} = -\frac{\mu}{\left(2\frac{K(t)}{\tilde{K}} + \sigma^2\right)^2}\left(2\frac{\partial_t K(t)}{\tilde{K}}\right), \tag{29}$$

which implies

$$\left|\frac{\partial_t \rho(t)}{\frac{\partial_t K(t)}{K(t)}}\right| = \frac{2\mu}{\left(2\frac{K(t)}{\tilde{K}} + \sigma^2\right)^2}\frac{K(t)}{\tilde{K}}. \tag{30}$$

The right-hand side of Eq. (30) is small as long as $\frac{K(t)}{\tilde{K}} \ll 1$, consistent with the expansion we performed.

Next we take the optimal $\rho(t)$ from above, and we study the implied stochastic differential equation for $K(t)$. For the case of $\gamma = 1$, we have

$$d\log(K/K_0) = \frac{1}{2}\frac{\mu^2}{2\mu\frac{K}{\tilde{K}} + \sigma^2}dt + \frac{\mu\sigma}{2\mu\frac{K}{\tilde{K}} + \sigma^2}dW. \tag{31}$$

When $K \gg \frac{\sigma^2\tilde{K}}{\mu}$, this simplifies to

$$\langle d\log(K/K_0)\rangle = \frac{1}{4}\frac{\mu\tilde{K}}{K}dt. \tag{32}$$

Since we observe that the asymptotic growth is compatible with a linear function of K, we can obtain the proportionality constant by using the ansatz $K(T) \approx aT$, and we obtain that the slope of the linear approximation is $\frac{\mu\tilde{K}}{4}$ for $T \gg \frac{\sigma^2}{4\mu^2}$. Further, when $\tilde{K} \to \infty$, this slope $\to \infty$ as well, because we are asymptotically approximating an exponential with a linear function.

2. $f(x) = \alpha\log(x)$

For the case of a logarithmic carrying capacity term, we have shown how to evaluate explicitly $\langle K(t)\rangle$ and $\langle\log[K(t)]\rangle$ in Eqs. (16). Using this solution, if we assume the stationary approximation in which ρ changes slowly compared to K, we obtain the following equation for the optimal parameter ρ in the limit $\delta t \to 0$:

$$\alpha\mu\left[\log\left(\frac{\tilde{K}}{\rho}\right) - \log[K(t)/K_0]\right] + \mu(1-\alpha) - \rho\sigma^2 = 0, \tag{33}$$

from which we can solve for $\rho(t)$:

$$\rho_{\mathrm{opt}}(t) \approx \alpha W\left(\frac{K_0 e^{\frac{1}{\alpha}-1}\tilde{K}\sigma^2}{\alpha K(t)\mu}\right)\frac{\mu}{\sigma^2}, \tag{34}$$

where W is the Lambert W function. Note that for $\alpha \to 0$, we recover again the result of [4]. This allows us to evaluate the critical ratio $\xi = \frac{K}{\tilde{K}}$ for which $\rho_{\mathrm{opt}} = 1$, which is given by $\xi = e^{\frac{1}{\alpha}(\frac{\sigma^2}{\mu}-1)-1}$.

Similarly to the case of a linear carrying capacity given in Eq. (31), we can obtain an effective differential equation by inserting the obtained optimal $\rho(t)$ of Eq. (34). Using the asymptotic properties of the Lambert W function in the limit $K \gg 1$, this differential equation is given by

$$\langle d\log(K/K_0)\rangle \approx \tilde{K}\mu\, dt, \tag{35}$$

[1]Such a calculation can be repeated in the presence of a risk-free asset with return, μ_{rf}. In this case, μ_{rf} would simply be added at the denominator of Eq. (28).

which implies an asymptotic linear growth given by $K(T) \approx 4aT$, where $a = \frac{\mu\tilde{K}}{4}$ is the slope obtained in Eq. (32). We thus have the result that in the case of logarithmic carrying capacity, the Kelly strategy implies a growth rate that is asymptotically twice the rate obtained for a linear function.

B. Analytical result: Short time scales

The approach of the previous section has some drawbacks. In particular, evaluating cumulants of the exponential of the Brownian motion is a lengthy task in general. While for the case of logarithmic carrying capacity it is possible to evaluate the averages exactly, this is not true in the general case. As an alternative, we can proceed by directly integrating the equations for the moments and elaborating an approximation scheme based on the smallness of the time horizon with respect to the other scales.

It is reasonable to expect that at least for small variations, the carrying capacity can be parametrized with a power series, leading to the following stochastic differential equation:

$$dK = \rho\mu K\left(1 + \sum_{k=1}^{n} \lambda_k(\rho K)^{k\gamma}\right)dt + \sigma\rho K\, dW. \tag{36}$$

As before, we focus on the maximization of the expectation value of the logarithm of K. Using Ito's lemma,

$$\left\langle \frac{d}{dt}\log K/K_0\right\rangle = \mu\rho\left(1 + \sum_{k=1}^{n}\lambda_k\rho^{k\gamma}\langle K^{k\gamma}\rangle\right) - \frac{\sigma^2\rho^2}{2}. \tag{37}$$

To solve this equation for generic values of the parameters λ_k, we need to compute all the moments $\langle K^{m\gamma}\rangle$, that is, we need to solve the equations of motion for these observables:

$$\left\langle \frac{d}{dt}K^{m\gamma}\right\rangle = \mu\rho\gamma m\left\langle\left(1 + \sum_{k=1}^{n}\lambda_k\rho^{k\gamma}K^{k\gamma}\right)K^{m\gamma}\right\rangle + \gamma m(\gamma m - 1)\frac{\sigma^2\rho^2}{2}\langle K^{m\gamma}\rangle. \tag{38}$$

These form a tower of coupled equations.[2] Using the notation $e_m := \langle K^{m\gamma}\rangle$, we can write the above as

$$\dot{e}_m = \mu\rho\gamma m\sum_{k=0}^{n}\lambda_k\rho^{k\gamma}e_{m+k} + \gamma m(\gamma m - 1)\frac{\sigma^2\rho^2}{2}e_m, \tag{39}$$

where $\lambda_0 = 1$. The initial conditions are $e_m(t_0) = K_0^{\gamma m}$, as the PDF for $K(t = t_0)$ is a Dirac δ at the initial time.

With these equations, given a time horizon δt, we can compute the Taylor expansion of the derivative $d\log K(t)/dt$ at any order in an expansion in δt. In general, this is given by

$$\frac{d}{dt}\log[K(t_0 + \delta t)] = \Phi(K_0,\mu,\sigma,\gamma,\{\lambda_k\};\delta t), \tag{40}$$

with

$$\Phi(K_0,\mu,\sigma,\gamma,\{\lambda_k = \delta_k^0\};\delta t) = \mu\rho - \frac{\sigma^2\rho^2}{2}. \tag{41}$$

Using this general procedure, the maximization needed to determine ρ is straightforward once a truncation in the expansion in δt has been fixed. As an example, consider the case in which

$$\gamma = 1, \quad n = 1, \quad \lambda_1 = -\frac{1}{\tilde{K}}. \tag{42}$$

To first order in δt, we then have

$$\left\langle \frac{d}{dt}\log K\right\rangle \simeq \mu\rho - \frac{\rho^2\sigma^2}{2} - \frac{K_0\mu\rho^2}{\tilde{K}} + \left(\frac{K_0^2\mu^2\rho^4}{\tilde{K}^2} - \frac{K_0\mu^2\rho^3}{\tilde{K}}\right)\delta t. \tag{43}$$

As expected, the corrections to the geometric Brownian motion case are controlled by the quantity $\frac{K_0}{\tilde{K}}$. With respect to these quantities, the optimal parameter ρ is

$$\rho_{\rm opt} \simeq \frac{\mu}{2\mu\frac{K_0}{\tilde{K}} + \sigma^2} + \frac{\left(-3\tilde{K}^3\mu^4\sigma^2K_0 - 2\tilde{K}^2\mu^5K_0^2\right)}{(2\mu K_0 + \tilde{K}\sigma^2)^4}\delta t. \tag{44}$$

In this last expression, we recognize, at zeroth order, the same term obtained in the approximation in which $\rho(t)$ is constant. At first order, we obtain a correction proportional to the size of the time horizon δt. These results can be generalized without difficulty to higher orders.

What is remarkable in the result of this short time-scale analysis is that the optimal value for ρ, in the general case, is a function of the initial condition, the parameters of the process *and* of the time horizon.

C. Numerical simulations

In this section, we present numerical tests of our analytical results, using a Monte Carlo approach and a stochastic Euler method to solve the differential equations. First, In Fig. 3 we report the expectation values of log-returns over a fixed time horizon $T = 1$ and for different values of the parameter ρ, assumed to be constant in time, and for different values for the scale of carrying capacity. Each point is an estimate for the expectation of the end point of the numerical solution of the corresponding stochastic differential equation. The curves that are obtained from these points clarify how the log-returns reach their maximum for special choices of ρ. The points on the upper curve are obtained in the case of no dependence on the amount of resources, i.e., the simple geometric Brownian motion. They match the results of [4] for the values of the parameters that we are considering. As expected, the plot shows that the optimal ρ decreases with the strength of carrying capacity.

In Fig. 4 we use Eq. (44) at zeroth order and compare this to the strategy at constant $\rho = 1$. We can see that the dynamic strategy outperforms those that are kept constant, and that it

[2]Formally, this tower can be rewritten in the form

$$\dot{\vec{e}} = M\vec{e},$$

with infinite-dimensional objects. A formal solution for given (time-independent) M is $e_m(t) = \sum_k R_{mk}(t)e_k(0)$, with R being the exponential of the operator M, $R(t) = e^{Mt}$.

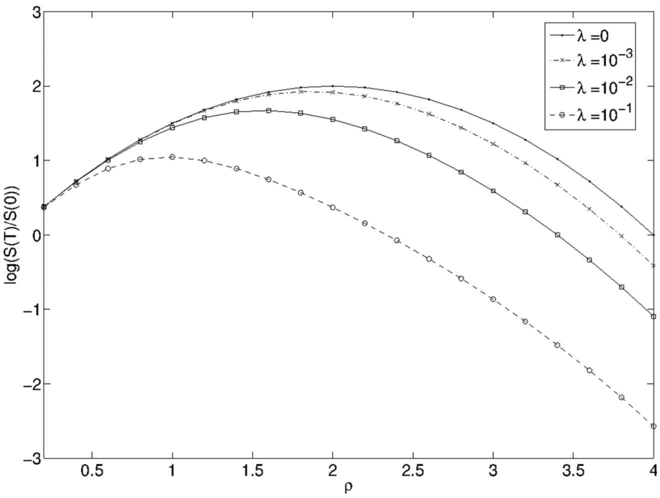

FIG. 3. Log return as a function of ρ at $T = 1$ for parameters $K(0) = 1$, $\mu = 2$, $\sigma = 1$, $\gamma = 1$, and various values of $\lambda = 1/\tilde{K}$.

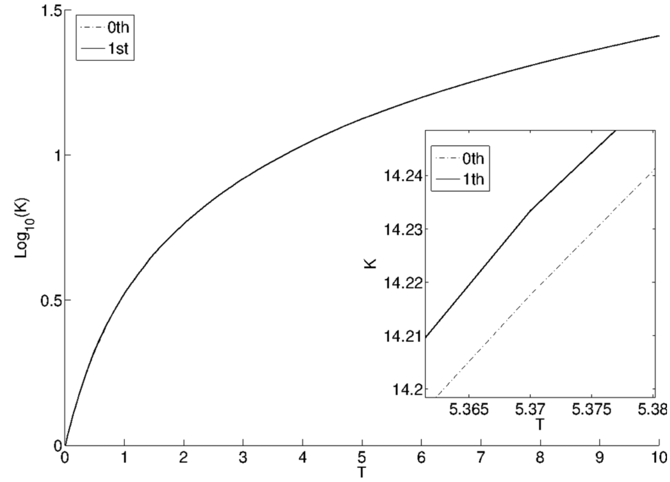

FIG. 5. Expected value of K for the case of power-law carrying capacity, with the optimal parameter ρ evaluated using the zeroth-order in the time horizon (dashed) and first-order (solid) correction for $K_0 = 1$, $\tilde{K} = 100$, $dt = 0.01$, $\sigma = 0.2$, $\mu = 1$, and $\gamma = 1$. To distinguish the two curves, we averaged over 1000 Monte Carlo runs. We observe that the solution obtained at first order outperforms the one obtained at zeroth order.

works reasonably well also in the case in which $K(t) \approx \tilde{K}$. In Fig. 5 we compare this result with the first-order approximation in δt. In the inset we see that the latter outperforms the optimal solution obtained at zeroth order, although the difference between the two is overall relatively small. For $\gamma \neq 1$, our solution also outperforms the constant solution.

For the logarithmic carrying capacity, we can see in Fig. 6 that our optimal parameter $\rho(t)$ solution again outperforms the constant solution. Also note that the stochastic equilibrium obtained from Eq. (12) is confirmed in both Figs. 4 and 6.

Finally in Fig. 7 we compare the linear regimes for the case of linear and logarithmic carrying capacity obtained in Eqs. (32) and (35) to the curves obtained with Monte Carlo simulations, showing that the slopes obtained analytically are a good match with the numerical ones.

IV. CONCLUSIONS

In this paper, we computed optimal strategies for the problem of maximal growth using the Kelly criterion in the case of a drift term that represents the presence of carrying capacity. Using two different methods, one considering exact solutions at constant leverage and the second solving for the optimal solution at a fixed time horizon, we obtained the same result at the lowest order. Our solutions were also tested numerically, confirming that these are optimal as compared to the case in which carrying capacity is ignored.

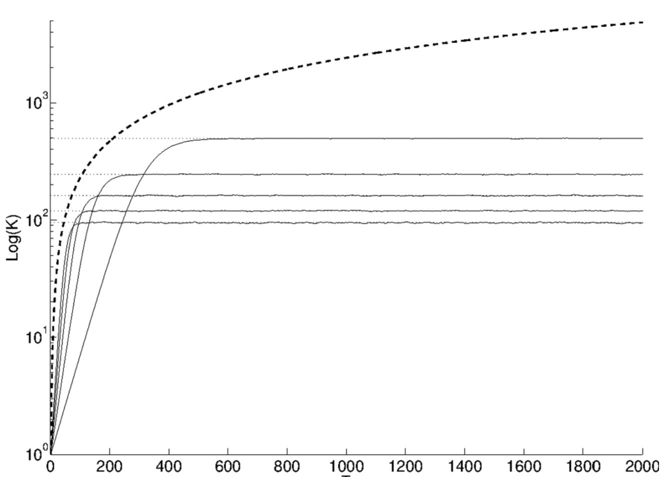

FIG. 4. Optimal Kelly path for K with carrying capacity function x^γ plotted against the case of those fixed at $\rho = 1, 0.8, 0.6, 0.4$, and 0.2 (full lines). The parameters used are $K(0) = 1$, $dt = 0.01$, $\mu = 0.1$, $\sigma = 0.1$, $\tilde{K} = 10$, and $\gamma = 1$, averaged over 2000 samples. The horizontal dashed lines represents the stochastic equilibria obtained from the analytic formula.

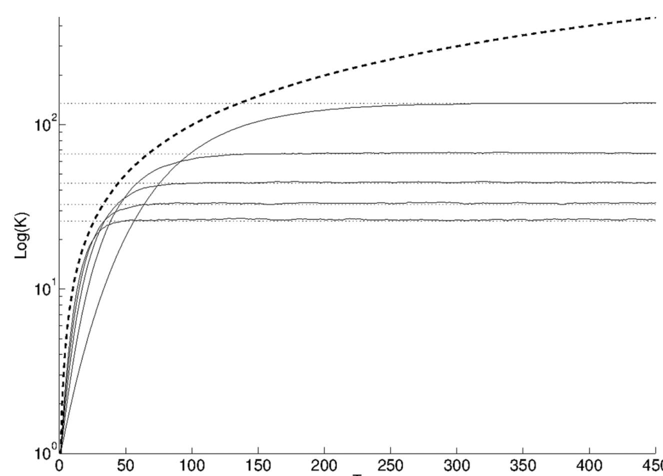

FIG. 6. Optimal Kelly time path of K for logarithmic carrying capacity compared to the case of those fixed at $\rho = 1, 0.8, 0.6, 0.4$, and 0.2 (full lines). The parameters used were $K(0) = 1$, $dt = 0.01$, $\mu = 0.1$, $\sigma = 0.1$, $\tilde{K} = 10$, and $\gamma = 1$, averaged over 2000 samples. The horizontal dashed lines represents the stochastic equilibria obtained from the analytic formula.

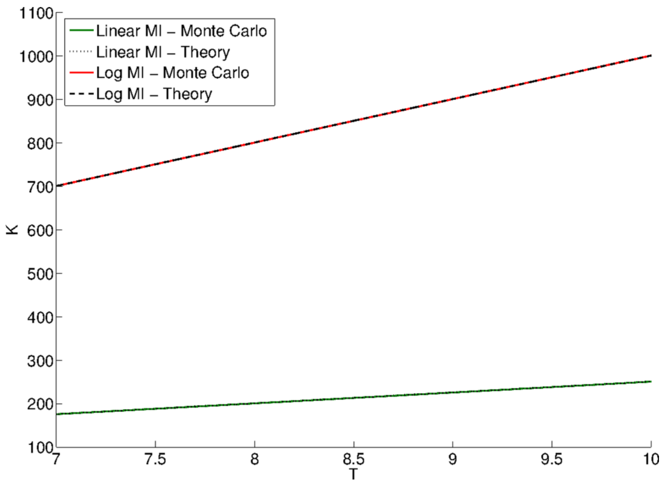

FIG. 7. Average asymptotic ($T \approx 7$) linear regime in the case of linear (green curve) and logarithmic (red curve) carrying capacity for the case of $\sigma = 0.2$, $\mu = 1$, and $\tilde{K} = 10$ obtained using a Monte Carlo averaged over 2000 samples, with integration step $dt = 0.01$. The dashed lines represent the comparison with the linear coefficients obtained from theory, Eqs. (32) and (35).

We considered two specific carrying capacity functions for which empirical evidence has been presented in the literature [21,22]: a power law and a logarithmic function. In the case of a power-law carrying capacity function, we have shown that in order to evaluate the optimal solution, various approximations have to be used, but we have shown that the zeroth order is correct up to second order in the parameter controlling the scale of the carrying capacity, and we provided a solution up to first order for the case of a finite time horizon. In the case of a logarithmic carrying capacity function, expectations can be evaluated exactly.

The main difference between the case of the geometric Brownian motion and the case with carrying capacity is that the former requires a constant leverage while in the latter the leverage has to be dynamically adapted. The operator has to adapt his strategy continuously depending on his position with respect to the capacity parameter. Our results support this intuitive observation and, at the same time, complement it with concrete procedures for quantitative estimates.

Finally, the analysis presented relies heavily on the Gaussian nature of the noise term and on the powerful results of Ito's calculus. Our results can then be seen as an assessment of the effects of carrying capacity on investment strategies, which, however, will require further elaboration. In particular, a natural extension will be the inclusion of more realistic noise terms, such as general Levy processes. The investigation of the impact of a more detailed noise structure on the optimal leverage will be the subject of future work.

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APPENDIX A: SOLUTION OF THE STOCHASTIC DIFFERENTIAL EQUATIONS FOR CONSTANT ρ

1. Power-law function

In this appendix, we discuss the solution of the equation

$$dK = \mu K(1 - K^{\gamma})dt + \sigma K\, dW_t. \tag{A1}$$

Using the change of variables $y = K^{-\gamma}$ and Ito's lemma, we have

$$dy = \left(-\gamma\mu(y-1) + \frac{\sigma^2\gamma(\gamma+1)}{2} y \right) dt - \gamma\sigma y\, dW_t. \tag{A2}$$

Notice that Eq. (A2) is a stochastic differential equation of the form

$$dz = (az + c)dt + (bz + d)dW, \tag{A3}$$

where

$$\begin{aligned} a &= \frac{\sigma^2\gamma(\gamma+1)}{2} - \gamma\mu, \\ b &= -\sigma\gamma, \\ c &= \gamma\mu, \\ d &= 0. \end{aligned} \tag{A4}$$

This is an inhomogeneous linear stochastic differential equation with multiplicative noise [20], and it has a known solution. If we define

$$\begin{aligned} \Phi_t &\equiv \exp[(a - b^2/2)t + bW_t] \\ &= \exp[-(\mu - \sigma^2/2)t - \sigma W_t], \end{aligned} \tag{A5}$$

the solution is then given by

$$z = \Phi_t \left(z_0 + (c - bd) \int_0^t \Phi_s^{-1} ds + d \int_0^t \Phi_s^{-1} dW_s \right). \tag{A6}$$

Writing

$$f_t(a,b,c,z_0) = \Phi_t \left(z_0 + c \int_0^t \Phi_s^{-1} ds \right), \tag{A7}$$

we obtain

$$y(t) = f_t \left(-\gamma\mu + \frac{\sigma^2\gamma(\gamma+1)}{2}, -\gamma\sigma, \gamma b, y_0 \right). \tag{A8}$$

Going back to the variable $K(t)$, we have

$$K(t) = \left[f_t \left(-\gamma\mu + \frac{\sigma^2\gamma(\gamma+1)}{2}, -\gamma\sigma, \gamma b, y_0 \right) \right]^{-1/\gamma}. \tag{A9}$$

If we examine the special case of $\gamma = 1$, and insert again the constants ρ and $\tilde{K}$ by rescaling $\mu \to \rho\mu$ and $\sigma \to \rho\sigma$, $\tilde{K} \to \tilde{K}/\rho$, we obtain the full solution in terms of all the original parameters:

$$\begin{aligned} K(t) = &\frac{\tilde{K}}{\rho} e^{(\rho\mu - \frac{(\rho\sigma)^2}{2})t + \rho\sigma W_t} \\ &\times \left(\frac{\tilde{K}}{\rho K_0} + \rho\mu \int_0^t e^{(\rho\mu - \frac{(\rho\sigma)^2}{2})s + \rho\sigma W_s} ds \right)^{-1}, \end{aligned} \tag{A10}$$

which is the solution used in this paper in the case of a power-law carrying capacity.

2. Logarithmic function

In the case of a logarithmic carrying capacity function, and again if $\rho(t) = \rho = 1$, we have the following differential:

$$dK = \mu K\left[1 - \alpha \log\left(\frac{K}{\tilde{K}}\right)\right]dt + \sigma K\, dW_t, \tag{A11}$$

which is a stochastic Gompertzian-type of equation [23,24]. Such an equation appears, for instance, also in the growth of reproducing cells, where now $\tilde{K}/\rho(t)$ represents the amount of nutrient accessible to the cells. This implies that there is a parallel between optimal leverage trajectories and optimal cell growth.[3] If we change variables to $y = \log(K)$, then through Ito's lemma the above becomes

$$dy = \left[\mu(1 - \alpha y + \alpha \log(\tilde{K})) - \frac{\sigma^2}{2}\right]dt + \sigma\, dW_t. \tag{A12}$$

This has the same form as Eq. (A3) in the previous section, with

$$\begin{aligned} a &= -\alpha\mu, \\ b &= 0, \\ c &= \mu[1 + \alpha \log(\tilde{K})] - \frac{\sigma^2}{2}, \\ d &= \sigma. \end{aligned} \tag{A13}$$

We then have that

$$\Phi_t = \exp(-\alpha\mu t), \tag{A14}$$

and thus one obtains

$$\begin{aligned} K(t) &= \exp\left\{e^{-\alpha\mu t}\left[C_0 + \left(\mu[1 + \alpha\log(\tilde{K})] - \frac{\sigma^2}{2}\right)\right.\right. \\ &\quad \left.\left.\times \int_0^t e^{\alpha\mu s} ds + \sigma \int_0^t e^{\alpha\mu s} dW_s\right]\right\} \\ &= \exp\left\{e^{-\alpha\mu t}\left[C_0 + \left(\mu[1 + \alpha\log(\tilde{K})] - \frac{\sigma^2}{2}\right)\right.\right. \\ &\quad \left.\left.\times \frac{e^{\alpha\mu t} - 1}{\alpha\mu} + \sigma \int_0^t e^{\alpha\mu s} dW_s\right]\right\}. \end{aligned} \tag{A15}$$

APPENDIX B: AVERAGE $\langle K \rangle$ FOR $f(x) = x$

In this appendix, we evaluate the average $\langle K \rangle$ as an expansion of $\xi = \frac{K}{\tilde{K}}$ of the denominator of the solution in Eq. (12), and we show that the optimal leverage obtained in Eq. (28) holds up to second order in ξ. For simplicity, we will set $\rho = 1$ during the calculation of the averages, and then restore $\rho \neq 1$ by rescaling $\sigma \to \rho\sigma$, $\mu \to \rho\mu$, and $\tilde{K} \to \tilde{K}/\rho$. In this case, expanding (11) to order $(K(t)/\tilde{K})^2$, we have that

$$\begin{aligned} \mathcal{K}(\delta t, \rho = 1, K(t)) &= \tilde{K} e^{(\mu - \frac{\sigma^2}{2})\delta t + \sigma W_{\delta t}} \\ &\quad \times \left(\frac{\tilde{K}}{K(t)} + \mu \int_0^{\delta t} e^{(\mu - \frac{\sigma^2}{2})s + \sigma W_s} ds\right)^{-1} \\ &\approx K(t)\, e^{(\mu - \frac{\sigma^2}{2})\delta t + \sigma W_{\delta t}} \\ &\quad \times \left(1 - \frac{\mu K(t)}{\tilde{K}} \int_0^{\delta t} e^{(\mu - \frac{\sigma^2}{2})s + \sigma W_s} ds\right). \end{aligned} \tag{B1}$$

Taking the expectation of the above, we have

$$\langle \mathcal{K} \rangle = \langle K_{\tilde{K}=\infty}(t) \rangle - \frac{\mu K(t)}{\tilde{K}} \langle K_{\tilde{K}=\infty}(t) F[W, \delta t] \rangle, \tag{B2}$$

with $F[W,t] = \int_0^t e^{(\mu - \frac{\sigma^2}{2})s + \sigma W_s} ds$ being the integral of an exponential Gaussian process, and $\mathcal{K}_{\tilde{K}=\infty} = K(t) e^{(\mu - \frac{\sigma^2}{2})\delta t + \sigma W_{\delta t}}$. Using $\langle e^{\sigma W_s} \rangle = e^{\frac{\sigma^2}{2}s}$, we then have that

$$\langle \mathcal{K}_{\tilde{K}=\infty} \rangle = K(t) e^{\mu\delta t}, \tag{B3}$$

and further using $\langle e^{\sigma(W_s + W_{s'})} \rangle = e^{\frac{\sigma^2[s + s' + 2\min(s,s')]}{2}}$ we get

$$\begin{aligned} \langle \mathcal{K}_{\tilde{K}=\infty}(t) F[W, \delta t] \rangle &= K(t) e^{\mu\delta t} \int_0^{\delta t} e^{(\mu + \sigma^2)s} ds \\ &= \frac{K_0}{\mu + \sigma^2} e^{\mu t}(e^{(\mu + \sigma^2)t} - 1). \end{aligned} \tag{B4}$$

Putting these together,

$$\begin{aligned} \langle \mathcal{K}(\delta t, \rho = 1, K(t)) \rangle &= K(t)\left[\left(1 - \frac{K(t)}{\tilde{K}} \frac{\mu}{\mu + \sigma^2}\right) e^{\mu\delta t}\right. \\ &\quad \left. - \frac{K(t)}{\tilde{K}} \frac{\mu}{\mu + \sigma^2} e^{(2\mu + \sigma^2)\delta t}\right] \\ &\quad + O\left(\frac{K_0}{\tilde{K}}\right)^2, \end{aligned} \tag{B5}$$

which is valid in the approximation $K_0 e^{\mu t} \ll \tilde{K}$. Restoring ρ, the above becomes

$$\begin{aligned} \langle \mathcal{K}(\delta t, \rho, K(t)) \rangle &= K(t)\left[\left(1 - \rho\frac{K(t)}{\tilde{K}} \frac{\mu}{\mu + \rho\sigma^2}\right) e^{\rho\mu\delta t}\right. \\ &\quad \left. - \rho\frac{K(t)}{\tilde{K}} \frac{\mu}{\mu + \rho\sigma^2} e^{(2\rho\mu + \rho^2\sigma^2)\delta t}\right] \\ &\quad + O\left(\frac{K_0}{\tilde{K}}\right)^2. \end{aligned} \tag{B6}$$

We now impose $\partial_\rho[\rho(t)\mu - \rho(t)^2(\mu\frac{\langle \mathcal{K}(\delta t, \rho, K(t)) \rangle}{\tilde{K}} + \frac{\sigma^2}{2})] = 0$, and after expanding at order $[K(t)/\tilde{K}]^2$ and imposing $\delta t \to 0$, we obtain again the equation for ρ:

$$\mu - \rho\sigma^2 - \frac{2\mu\rho K(t)}{\tilde{K}} = 0. \tag{B7}$$

This implies that the optimal leverage obtained at zeroth order is valid up to second order in $K(t)/\tilde{K}$. In general, it is possible to use perturbation theory to obtain higher-order approximations of this result, using, for instance, the exact formulas obtained in [25].

[3] It is interesting to note that in general, the logarithmic carrying capacity function can be thought of as the asymptotic limit of a power-law function, as one has $\lim_{\alpha \to \infty} \alpha(1 - x^{\frac{1}{\alpha}}) = -\log(x)$.

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